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ANALYTICAL SOLUTION OF A NON - HOMOGENEOUS ONE - DIMENSIONAL ADVECTION DIFFUSION EQUATION WITH TEMPORALLY VARYING COEFFICIENTS

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ABSTRACT
Advection Diffusion Equation is a partial differential equation that describes the transport of pollutants in rivers. Its coefficients (dispersion and velocity) can be constant, dependent on space or time or both space and time. This study presents an analytical solution of a one dimensional non - homogeneous advection diffusion equation with temporally dependent coefficients, describing one dimensional pollutant transport in a section of a river. Temporal dependence is accounted for by considering a temporally dependent dispersion coefficient along an unsteady flow assuming that dispersion is proportional to the velocity. Transformations are used to convert the time dependent coefficients to constant coefficients and to eliminate the advection term. Analytical solution is obtained using Fourier transform method considering an instantaneous point source. Numerical results are presented. The findings show that concentration monotonically decreases with increasing distance and increasing time.

KEYWORDS: Temporal Variation, Instantaneous Point Source.

1. INTRODUCTION
In recent years general interest in preserving the quality of water has tremendously increased. Many problems relating to quality of water in the rivers is as a result of pollution. Many rivers are recipients and carriers of pollutants from all sources including industries, untreated sewers, pesticides from agricultural activities, uncollected garbage from human beings e. t. c. The managers therefore need reliable support tools for assessment of distribution of pollutant concentration to aid them in decision making. This issue can be addressed using computational tools such as mathematical models that can provide changes in concentration with respect to space and time. One of such models, that describe the transport of pollutants in rivers, is the Advection Diffusion Equation (ADE). ADE is a parabolic partial differential equation based on conservation of mass and Fick's first law. Its coefficients may be constant or may depend on space, time or both time and space. Its solution can be obtained analytically or numerically. Analytical solutions helps in the understanding of the mechanism of pollutant transport, provides initial or approximate analysis of alternative pollution scenarios and acts as a benchmark for the validation of numerical codes. These solutions subject to various initial and boundary conditions, depicting different real life situations, are also useful in assessing the position and time in which the concentration level of a pollutant will start affecting human and aquatic health.

Analytical solutions for an ADE with constant, spatial and temporal coefficients, subject to different initial and boundary conditions have been determined by several authors. Basha and El – Habel (1993) obtained an analytical solution of the ADE with temporally dependent dispersion coefficient and uniform velocity, considering instantaneous and continuous mass injection of the pollutant. The temporally dependent dispersion coefficient assumed linear, asymptotic and exponential functional forms. The dispersion coefficient was dependent on the travel time of the pollutant from a single input source. Kumar *et al* (2010) used Laplace transforms to solve the 1D ADE with variable coefficients for three dispersion problems: solute dispersion along steady flow through inhomogeneous medium, temporally dependent solute dispersion along uniform flow through homogeneous medium and solute dispersion along temporally dependent flow through inhomogeneous medium. They considered a continuous input point source of uniform and increasing nature in an initially solute free semi –

infinite medium. The dispersion parameter was considered proportional to the square of the spatially dependent velocity.

Jaiswal *et al* (2011), also used Laplace transforms to determine an analytical solution of the 1D ADE with both velocity and dispersion time dependent. They studied a case of continuous input point sources of uniform and increasing nature in an initially solute free semi - infinite domain. Kumar *et al* (2012) extended the work of Kumar *et al* (2010), by solving an ADE with spatial - temporal dispersion and velocity coefficients, a case where the dispersion parameter is proportional to the square of the velocity. They presented results of exponentially increasing and exponentially decreasing functions of time. Dimian *et al (2013)* used Laplace transforms to determine the analytical solution for one dimensional ADE with variable coefficients in a longitudinal finite initially pollutant free domain. They also considered boundary conditions as continuous input point sources of uniform and increasing nature. By writing the equations in the dimensionless form, they reduced the five physical parameters controlling the pollutant concentration to only two dimensionless parameters: the dimensionless added pollutant concentration and the dimensionless dispersion. They found that some physical parameters in the dimensional form have the same effect on the concentration of the pollutant, while other physical parameters have opposite effect. Their study concluded that the dimensionless pollutant concentration increases as the dimensionless added pollutant increases along the river. But the concentration decreases, as the dimensionless dispersion increases. Kumar (2017) presented analytical solutions for a 1D ADE with temporally dependent variable coefficients of hyperbolic function in semi - infinite porous domain. Unlike the previous works by(Kumar *et al* (2010), Jaiswal *et al* (2011), Kumar *et al* (2012), Dimian *et al (2013)* and Kumar (2017)), where the medium was assumed to be initially solute free; Yadav and Kumar (2018), studied a 1D ADE with spatially dependent dispersion and velocity coefficients assuming that the medium is initially not solute free. In this study, they considered varying type input condition for multiple point sources of arbitrary time - dependent emission rate pattern at the origin. In similar studies, Yadav and Jaiswal (2011) varied the dispersion coefficient spatially with seepage velocity exponentially decreasing function of space, in a study on a 2D ADE with a constant input concentration along unsteady horizontal flow in a semi - infinite shallow aquifer.

Exact solutions of the ADE with temporally dependent dispersion and spatial - temporal velocity in an infinite domain subjected to instantaneous and continuous injection was obtained by Sanskrityayn and Kumar (2016) using Greens function method. A similar study was done by Pintu *et al* (2017). For each of these two problems, (i.e. temporally dependent dispersion and spatial - temporal velocity), Pintu and colleagues varied the velocity patterns as exponentially decreasing functions, sinusoidally varying function and algebraic sigmoid functions. Greens function method was also used by Park and Zhan (2001) to solve a contaminant transport from a finite 1D, 2D and 3D sources (point, line and area) in a finite thickness. They noted that the concentration in the near field is sensitive to the source geometry and anisotropy of the dispersion coefficients. The contaminant concentration in the field was found to be much less sensitive to the source geometry. Wadi *et al* (2014) solved the 1D unsteady ADE by using Laplace Transforms technique. The river was divided into two regions: upstream $x < 0$ near the source where it was assumed that the rate of pollutant addition along the river vanishes and the downstream region $0 < x < L$ where the rate of pollutant addition along the river is constant. They solved the ADE while considering both the half - saturated oxygen demand concentration for pollutant decay and the dissolved oxygen within the river in both regions.

Mazaheri *et al* (2013) presented exact solutions for an ADE with constant velocity and diffusion coefficient for point source with a linear pulse time pattern and extended the domain for an arbitrary time pattern involving several point sources by using Laplace transforms. They observed that the analytical solution obtained from this study is suitable for cases where the ADE is solved over large temporal or spatial intervals or in cases where the solution needs to be estimated only in some specific points, and cases where concentration to constant parameter conditions are required. Mojtabi and Deville (2015) made a comparison between analytical and numerical solutions of a 1D ADE. They employed separation of variables method to solve the resulting heat equation, subject to dirichlet homogeneous boundary condition and an initial sine function, after using a change of variable in the original ADE. The studies done by Guerrero and Skaggs (2010) and Subhrangshu and Kumar (2018) focused on the analytical solution of a 1D ADE with spatially dependent velocity and dispersion coefficients. The latter considered time varying boundary conditions and used the method of Eigen function expansion while the former used the Generalized Integral Transformation Technique (GITT) to solve the resulting transformed equations.

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Sanskrityayn *et al* (2016) used Green's function method to determine an analytical solution of 1D ADE with spatial - temporal velocity and dispersion coefficients, modeling pollutant mass transport in a heterogeneous medium originating from an instantaneous source. Green's function method was equally used by Xu *et al* (2007) to solve the 1D ADE in infinite, semi - infinite and finite domains with Dirichlet, Neumann and Robin boundary conditions.

A number of studies have been conducted on determination of analytical solutions of 1D ADE considering the ADE with constant coefficients, spatially dependent and time dependent coefficients. Some of these studies considered uniform input point source and input point source of increasing nature while others considered instantaneous point source and continuous point source. Some of the studies that considered instantaneous and continuous point source include Basha and El – Habel (1993) who studied time dependent dispersion of the ADE and a uniform velocity, Sanskrityayn *et al* (2016) who extended the work of Basha and El – Habel (1993) by considering a temporally dependent dispersion coefficient and a spatial - temporal velocity coefficient and; Sanskrityayn and Kumar (2016) who considered a spatial - temporal velocity and dispersion coefficients. The present work considers velocity and dispersion coefficients temporally varying, a case where the pollutant dispersion is directly proportional to the flow velocity. Temporally dependent ADE with instantaneous point source is transformed into an ADE with constant terms, and transformed further to a diffusion problem. Fourier transform is then used on the resulting equation to obtain the solution.

2. MATERIALS AND METHODS

2.1 Model Formulation

Consider a 1D linear Advection Diffusion Equation describing pollutant transport in a river of infinite length $-\infty \leq x \leq \infty, t > 0$. The general linear form of one - dimensional advection diffusion equation, describing transport of pollutants in a river, in Cartesian form is given by:

$$\frac{\partial C}{\partial t} - D(x,t)\frac{\partial^2 C}{\partial x^2} + U(x,t)\frac{\partial C}{\partial x} = S(x,t) \quad (1)$$

Where C(x, t) is the pollutant concentration, U(x, t) and D(x, t) is the flow velocity and the dispersion coefficients respectively while S(x, t), the source term representing non - dimensional instantaneous injection of a pollutant mass, is given by:

$$S(x,t) = S_0\delta(x)\delta(t) \quad (2)$$

S_0 is the injected mass and $\delta(.)$ is the Dirac delta function. Also given the temporally dependent dispersion along unsteady flow, a case where the dispersion coefficient is proportional to the velocity, we have

$$D(x,t) = D_0 f(mt) \text{ and } U(x,t) = U_0 f(mt) \quad (3)$$

where D_0 and U_0 are initial dispersion and uniform velocity coefficients respectively in a homogeneous medium, $f(mt)$ is a non - dimensional expression, m is the unsteady parameter whose dimension is inverse of that of the time variable t. Rewriting (1) using (3) yields

$$\frac{\partial C}{\partial t} - D_0 f(mt)\frac{\partial^2 C}{\partial x^2} + U_0 f(mt)\frac{\partial C}{\partial x} = S(x,t) \quad (4)$$

Since we are considering an initially pollutant free domain, equation (4) is solved subject to the initial condition

$$C(x,0) = 0; -\infty \leq x \leq \infty \quad (5)$$

2.2 Analytical Solution

To obtain the solution of temporally dependent dispersion and velocity coefficients ADE (4), transformations are used to reduce the variable coefficients to constant coefficients. The constant coefficients ADE are then reduced to a diffusion equation using other transformations, and Fourier transforms finally used to determine the solution

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to the resulting diffusion equation. Previously applied transformations are re – applied to obtain the solution of the ADE with temporally varying coefficients.

We first transform (4) into an ADE with constant coefficients using the transformations given by Crank (1956).

$$T(t) = \int_0^t f(m\tau)d\tau \qquad (6)$$

Where m is chosen such that for m = 0 or t = 0, we get T = 0. Using (6) in (4) we get an ADE with constant diffusion and advection coefficients in the new time variable T:

$$\frac{\partial C}{\partial T} - D_0\frac{\partial^2 C}{\partial x^2} + U_0\frac{\partial C}{\partial x} = \frac{S(x,T)}{f(mt)}\ ; -\infty \le x \le \infty, T > 0 \qquad (7)$$

Next we eliminate the advection term from equation (7) using the transformations advanced by Jaiswal *et al* (2011)

$$C(x,T) = Q(x,T)\exp\left(\frac{U_0 x}{2D_0} - \frac{{U_0}^2 T}{4D_0}\right) \qquad (8)$$

Differentiating (8) once with respect to T, once and twice with respect to x, and inserting the results in (7) yields

$$\frac{\partial Q}{\partial T} = D_0\frac{\partial^2 Q}{\partial x^2} + S_1(x,T) \qquad (9)$$

$$Q(x,0) = 0; -\infty \le x \le \infty \qquad (10)$$

$$S_1(x,T) = \frac{S(x,T)}{f(mt)}\exp\left(\frac{{V_0}^2 T}{4D_0} - \frac{V_0 x}{2D_0}\right) \qquad (11)$$

Equation (10) and (11) is the initial condition and source term respectively. Equation (9) is a non – homogeneous diffusion equation with constant diffusion coefficient D_0. Applying Fourier transforms on (9) yields a linear ODE

$$\frac{d\hat{Q}(\omega,T)}{dT} + D_0\omega^2\hat{Q}(\omega,T) = \widehat{S_1}(\omega,T); \qquad (12)$$

Where $\hat{Q}(\omega,T) = \int_{-\infty}^{\infty} Q(x,T)e^{-i\omega x}\,dx$

Applying Fourier transforms on the initial condition (10) gives

$$\hat{Q}(\omega,0) = 0; -\infty \le \omega \le \infty \qquad (13)$$

Multiplying (12) by the integrating factor $I.F = e^{\int D_0\omega^2 dT} = e^{D_0\omega^2 T}$ and integrating the result from 0 to T yields

$$e^{D_0\omega^2 T}\hat{Q}(\omega,T) = \hat{Q}(\omega,0) + \int_0^T e^{D_0\omega^2\tau}\widehat{S_1}(\omega,\tau)\,d\tau$$

Applying (13)

$$\hat{Q}(\omega,T) = \int_0^T e^{-D_0\omega^2(T-\tau)}\widehat{S_1}(\omega,\tau)\,d\tau \qquad (14)$$

According to Konstantinos (2005), $F[G(x,T-\tau)] = \hat{G}(\omega,T-\tau) = e^{-D_0\omega^2(T-\tau)}$. Rewriting (14) in terms of the Gaussian distribution G (.,.)

$$\hat{Q}(\omega,T) = \int_0^T \hat{G}(\omega,T-\tau)\widehat{S_1}(\omega,\tau)\,d\tau \qquad (15)$$

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Finding inverse on both sides of (15)

$$Q(x,T) = \int_0^T F^{-1}[\hat{G}(\omega, T - \tau)\widehat{S_1}(\omega,\tau)]d\tau \quad (16)$$

giving us a convolution in the spatial domain since equation (16) has a product of two Fourier transforms.

Definition 1{Convolution (Echols (2015))}: The convolution of two functions f(x) and g(x) is the function $f * g$ defined by $f * g = \int_{-\infty}^{\infty} f(x-s)g(s)ds$

Definition 2 {Convolution Theorem (Echols (2015))}: Let f(x) and g(x) be two functions with convolution$f * g$, then $F[f * g] = \hat{f}\hat{g}$ where $\hat{g}$ and $\hat{f}$ are Fourier transforms of g(x) and f(x) respectively.

By definition 2 followed by definition 1:

$$F^{-1}[\hat{G}(\omega, T - \tau)\widehat{S_1}(\omega,\tau) = G * S_1 = \int_{-\infty}^{\infty} G(x-s, T-\tau)\, S_1(s,\tau)d\tau \quad (17)$$

Substituting (17) in (16)

$$Q(x,T) = \int_0^T \int_{-\infty}^{\infty} G(x-s, T-\tau)S_1(s,\tau)dsd\tau \quad (18)$$

Since $G(x,T) = F^{-1}[\hat{G}(\omega,T)] = \frac{1}{\sqrt{4\pi D_0 T}} \exp\left\{\frac{-x^2}{4D_0T}\right\}$ according to Konstantinos (2005)

Then

$$G(x-s, T-\tau) = \frac{1}{\sqrt{4\pi D_0(T-\tau)}} \exp\left\{\frac{-(x-s)^2}{4D_0(T-\tau)}\right\} \quad (19)$$

Inserting (19) in (18)

$$Q(x,T) = \int_0^T \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi D_0(T-\tau)}} \exp\left\{\frac{-(x-s)^2}{4D_0(T-\tau)}\right\} s_1(s,\tau)\, dsd\tau \quad (20)$$

Using $s_1(s,\tau)$ from (11) and evaluating the integrals in (20), taking note of the shifting property of delta function, equation (20) reduces to

$$Q(x,T) = \frac{s_0}{f(mt)\sqrt{4\pi D_0 T}} \exp\left(\frac{-x^2}{4D_0T}\right) \quad (21)$$

Substituting (21) in (8) gives us the analytical solution of our ADE with constant dispersion and velocity coefficients. Finally, we use transformation (6) to give the desired analytical solution of an ADE with temporally dependent dispersion and advection coefficients.

$$C(x,t) = \frac{s_0}{f(mt)\sqrt{4\pi D_0 T}} \exp\left(\frac{-x^2}{4D_0T}\right)\exp\left(\frac{U_0x}{2D_0} - \frac{{U_0}^2T}{4D_0}\right) \quad (22)$$

3. RESULTS AND DISCUSSION

Numerical results are presented for a case of instantaneous point source with the dispersion and velocity coefficients exponentially increasing with time i.e. $f(mt) = e^{mt} = 1 + mT$ for $m > 0$ using (6). Then from (22)

$$C(x,t) = \frac{s_0}{(1+mT)\sqrt{4\pi D_0 T}} \exp\left(\frac{-x^2}{4D_0T}\right)\exp\left(\frac{U_0x}{2D_0} - \frac{{U_0}^2T}{4D_0}\right) \quad (23)$$

Where

$$T = \frac{1}{m}(e^{mt} - 1)$$

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Concentration profiles are obtained from (23) in a domain $x \in 1000m$ using MATLAB. Concentration distribution at different times t = 5, 10, 15 days using the unsteady parameter m = 0.1/day, $S_0 = 1\text{kg/m}^2$, $U_0 = 1.14\text{m/day}$, and $D_0 = 1.25\text{m}^2/\text{day}$ is given in figure 1.

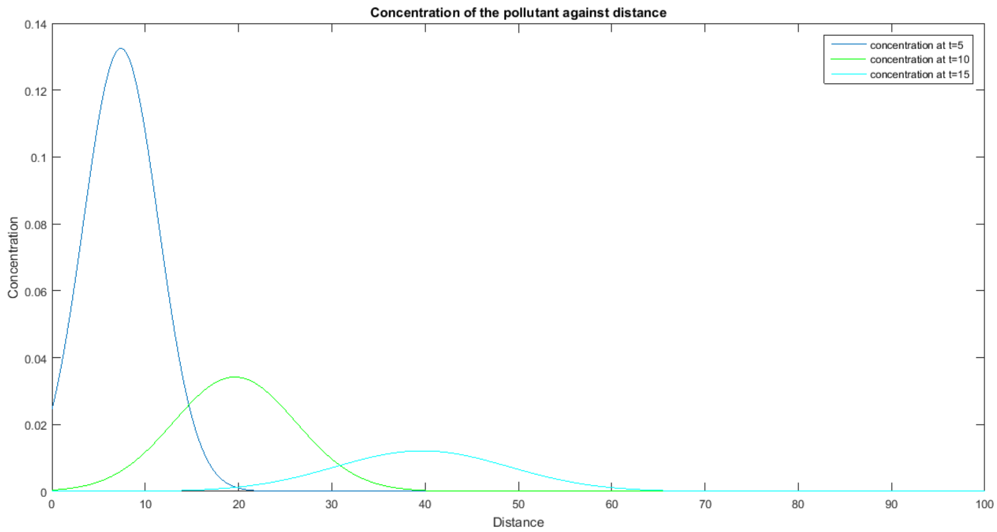

Figure 1: Comparison of pollutant concentration distribution at various times

Concentration is found to be higher at shorter times and lower after a long duration of time. The decrease in concentration with increasing time is attributed to the process of diffusion that spreads the pollutant molecules around the injection point and further; and the advection process that transports these molecules downstream. The decrease in pollutant concentration as the pollutant moves further away from the source acts as an indicator to water quality professionals that water in some a far distance from the source may be much safer for human consumption than those nearer the point of injection.

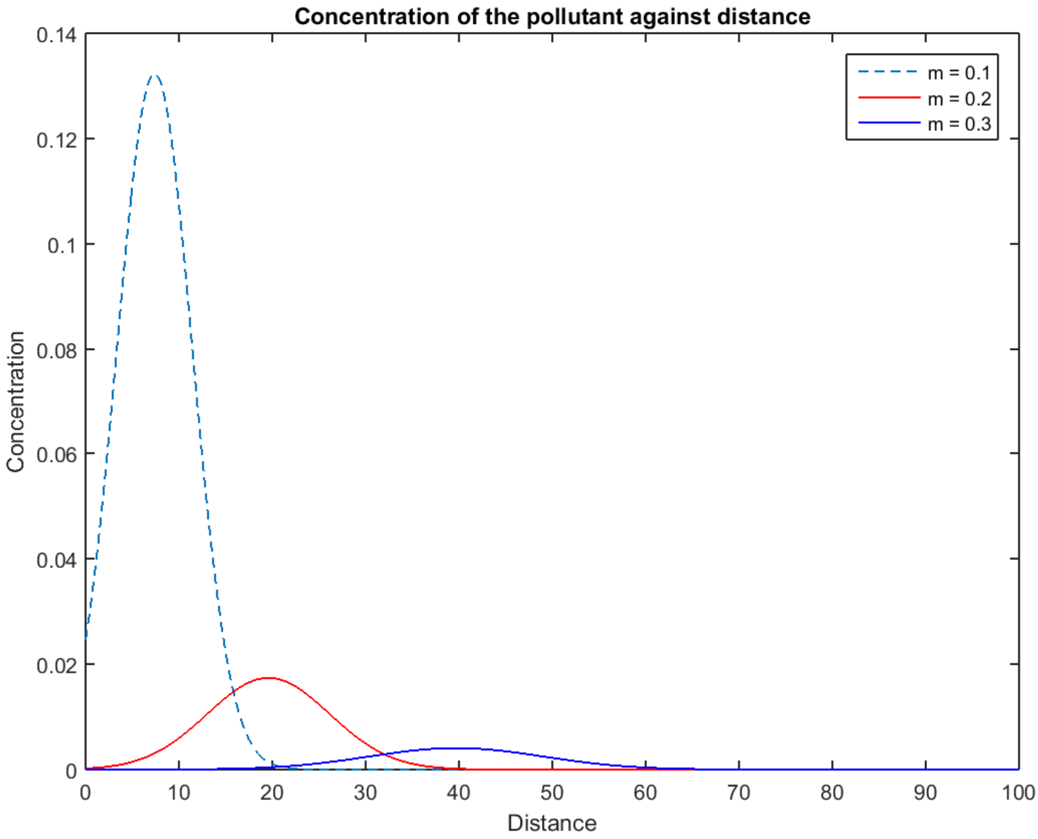

Figure 2: Comparison of pollutant concentration at a particular time for various unsteady parameters

Figure 2 depicts the concentration profile for various m at a particular time. This profile helps us understand the concentration behavior of the pollutant for various unsteady parameters m. Concentration distribution is given for dispersion parameter $D_0 = 1.25\text{m}^2/\text{day}$, flow velocity $U_0 = 1.14\text{m}/\text{day}$, the unsteady parameters m = 0.1, 0.2, 0.3 /day, constant mass input $S_0 = 1\text{kg}/\text{m}^2$ and fixed time t = 10 days. Concentration is found to be higher for a smaller unsteady parameter and lower for a larger unsteady parameter.

The spreading and transport of pollutants in rivers is always due to a combined effect of dispersion and advection processes. The study considered varying dispersion parameters against a constant velocity, to determine the contribution made by the dispersion process in pollutant concentration. Figure 3 gives the concentration profile for various dispersion parameters at a particular time and flow velocity. Concentration distribution is given for dispersion parameters; $= 1.25,\ 1.45,\ 1.65\text{m}^2/\text{day}$; the unsteady parameter m = 0.1/day, constant mass input$S_0 = 1\text{kg}/\text{m}^2$, fixed time t = 5 days and flow velocity $V_0 = 1.14\text{m}/\text{day}$. The dispersion parameter is found to contribute to the increase in pollutant concentration at a given position and a fixed time. At a particular time and distance, concentration is also found to be higher for a lower initial diffusion coefficient and lower for a higher initial diffusion coefficient. So if a low dispersion parameter is used, we are likely to get a high pollutant concentration.

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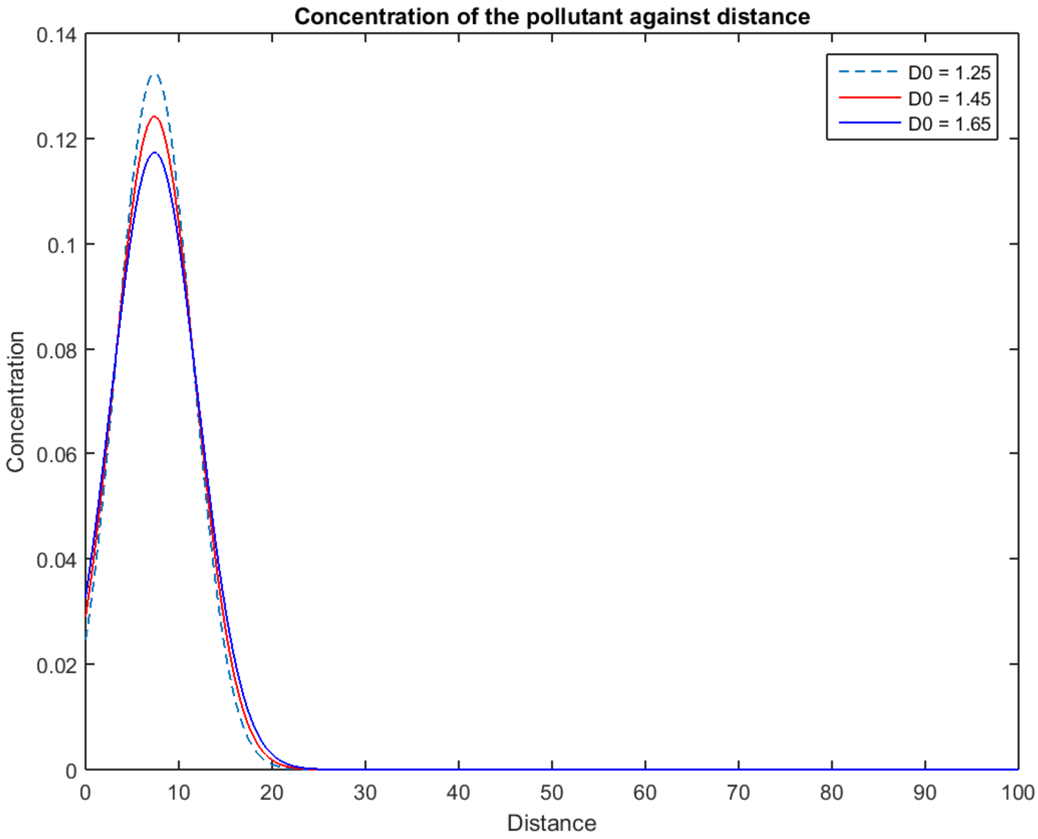

Figure 3: Comparison of pollutant concentration for various dispersion parameters at a particular time and flow velocity

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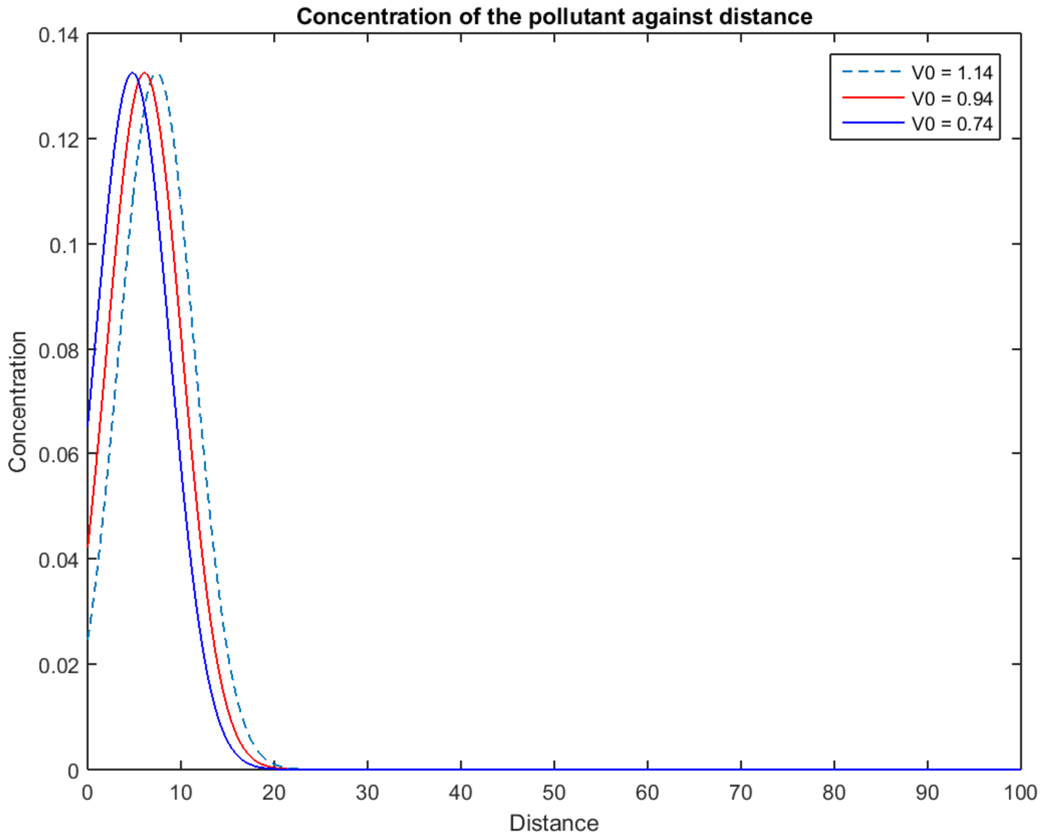

Figure 4: Comparison of pollutant concentration for various velocity coefficients at a particular time and dispersion parameter

Concentration profile in Figure 4 helps to understand the effect of the flow velocity in concentration distribution of a pollutant. It aims at comparing pollutant concentration for various velocity coefficients at a particular time and dispersion parameter. It is obtained using velocity coefficients; $V_0 = 1.14, 0.94, 0.74 \mathrm{m/day}$; the unsteady parameter m = 0.1/day, constant mass input$S_0 = 1 \mathrm{kg/m^2}$, fixed time t = 5 days and diffusion parameter $D_0 = 1.25 \mathrm{m^2/day}$. Velocity parameter is found to be responsible for the concentration attenuation downwards along the spatial axis. This account for the spatial variation in the profile and lack of change in pollutant concentration level, at a particular position and time, around the source position (see figure 4). At the injection point, concentration level is however higher for a smaller initial velocity coefficient and lower for a bigger initial velocity coefficient.

4. CONCLUSION

In this study analytical solution of a one dimensional ADE with temporally varying dispersion and velocity coefficients, a case where pollutant dispersion is proportional to flow velocity, is presented. It is observed that concentration generally decreases with increasing distance and decreases with increasing time. The increase in concentration level is as a result of the diffusion of pollutant molecules immediately the pollutant is introduced in the rivers. This process of diffusion together with the effect of advection is responsible for not only an increase in the concentration level around the point of injection but also for transporting the pollutant molecules downstream. Time dependent behavior of pollutants in rivers is very important in many real life situations where concentration need to be predicted at a particular time. Besides, with the provided information on the concentration distribution in space and time, water quality professionals are able to make the best decisions to counteract pollution.

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REFERENCES

[1] H. Basha, and S. El – Habel, "Analytical Solution of the 1D Time Dependent Transport Equation," in Water Resources Research, Vol. 29, No.9, pp. 3209 - 3214, 1993.
[2] J. Crank, "Mathematics of Diffusion," Oxford University Press, New York, 1956.
[3] M. Dimian, A. Wadi and F. Ibrahim, "The Effect of Added Pollutant along a river on the Pollutant Concentration described by One – Dimensional Advection Diffusion Equation," International Journal of Engineering Science and Technology, Vol. 5, No. 9, pp. 1662-1671, Sep 2013.
[4] J.A. Echols, "Closed Form Solutions of the Advection Diffusion Equation via Fourier Transforms," Arizona State University, School of Electrical, Computing and Energy Engineering, APM 526, 2015.
[5] P. Guerrero and H. Skaggs, "Analytical Solution for 1D Advection Dispersion Transport Equation with Distance Dependent Coefficients," in Journal of Hydrology, Vol. 390, pp. 57 – 65, 2010.
[6] D.K. Jaiswal, A. Kumar and R.R. Yadav, "Analytical Solution to the 1D Advection Diffusion Equation with Temporally Dependent Coefficients," in Journal of Water Resources and Protection, Vol. 3, pp. 76 – 84, 2011.
[7] G. Konstantinos, "Fourier Transformation of the Gaussian," Derparis, 2005.
[8] A. Kumar, J. Kumar and N. Kumar, "Analytical Solutions of 1D ADE with Variable Coefficients in Semi - infinite Medium," in Journal of Hydrology, Vol. 380, pp. 330 – 337, 2010.
[9] A. Kumar, J. Kumar and R.R. Yadav, "Analytical Solutions of 1D Temporally Dependent ADE along Longitudinal Semi - infinite Homogeneous Porous Domain for Uniform Flow, in IOSR Journal of Mathematics, Vol. 2, No. 1, pp. 1 – 11, 2012.
[10] L.K. Kumar, "An Analytical approach for 1D ADE with Temporally Dependent Variable Coefficients of Hyperbolic Function in Semi - infinite Porous Domain," in Journal of Engineering Science and Technology, Vol. 4, No. 9, 2017.
[11] M. Mazaheri, J.M.V. Samani and H.M.V. Samani, "Analytical Solution to the Advection Diffusion Equation with Several Point Sources through Arbitrary Time - Dependent Emission Rate Patterns" in J. Agr. Sci. Tech., Vol. 15, pp. 1231 – 1245, 2013.
[12] A. Mojtabi and M. Deville, "ID Linear ADE: Analytical and Finite Element Solutions," Computers and Fluids, Elsevier, Vol. 107, pp. 189 – 195, 2015.
[13] E. Park and H. Zhan, "Analytical Modeling of Contaminant Transport and Horizontal Well Hydraulics," Journal of Contaminant Hydrology, Vol. 53, No. 1 – 2, pp. 41 – 61, 2001.
[14] D. Pintu, B. Sultana and M.S. Kumar, "Mathematical Modeling of Groundwater Contamination with Varying Velocity Field," J. Hydrol. Hydromech, Vol. 65, No. 2, pp.192 – 204, 2017.
[15] A. Sanskrityayn, V. Bharati and N. Kumar, "Analytical solution of Advection Diffusion Equation with Spatio - temporal Dependence of Dispersion Coefficient and Velocity using Green's Function Method," in Journal of Groundwater Research, Vol. 5, No. 1, 2016.
[16] A. Sanskrityayn and N. Kumar, "Analytical Solution of Advection Diffusion Equation in Heterogeneous Infinite Medium using Green's Function Method," in J. Earth. Syst. Sci., Vol. 125, No. 8, pp. 1713 – 1723, 2016.
[17] P. Subhrangshu and B. Kumar, "Analytical solution of the 1D Contaminant Transport Equation in Groundwater with Time varying Boundary Conditions," in ISH Journal of Hydraulic Eng., 2018.
[18] A. Wadi, M. Dimian and F. Ibrahim, "Analytical Solution for 1D Advection Diffusion Equation of the Pollutant Concentration," J. Earth. Syst. Sci., Vol. 123, No. 6, pp. 1317 – 132, 2014.
[19] Z. Xu, J. Travis and W. Breitung, "Green's Function Method and its Application to Verification of Diffusion Models of GASFLOW Code," Forschungszentrum Karlsruhe GmbH, Karlsruhe, 2007.
[20] R. Yadav and D.K. Jaiswal, "2D Analytical Solutions for Point Source Contaminants Transport in Semi - infinite Homogeneous Porous Medium," in Journal of Engineering Science and Tech, Vol. 6, No.4, pp. 459- 468,2011.
[21] R. Yadav and Kumar. L.K, "Analytical Solution of 1D ADE with Dispersion Coefficients as Function of Space in a Semi - infinite Porous Media," in Pollution, Vol. 4, No. 4, pp. 745 – 758, 2018.